Problem 8) a) Let $z_1(t)=e^{\eta t}$. Putting this function into the homogeneous equation yields: $\eta^2+\gamma\eta+\omega_0^2=0 \Rightarrow \eta_{1,2}=-\frac{\gamma}{2}\pm\sqrt{\frac{\gamma^2}{4}-\omega_0^2}$. With reference to Problem 7), we now write the homogeneous solution as follows:

Case i) $\gamma<2\omega_0$: $z_1(t)=Ae^{-(\frac{1}{2}\gamma-i\sqrt{\omega_0^2-\gamma^2/4})t}+Be^{-(\frac{1}{2}\gamma+i\sqrt{\omega_0^2-\gamma^2/4})t}$

Case ii) $\gamma=2\omega_0$: $z_1(t)=(A+Bt)e^{-\frac{1}{2}\gamma t}$

Case iii) $\gamma>2\omega_0$: $z_1(t)=Ae^{-(\frac{\gamma}{2}-\sqrt{\gamma^2/4-\omega_0^2})t}+Be^{-(\frac{\gamma}{2}+\sqrt{\gamma^2/4-\omega_0^2})t}$.

b) The particular solution for $t>0$ may be written $z_1(t)=Ce^{i2\pi f_0 t}$. Substitution into the differential equation yields:

$$(i2\pi f_0)^2 C\cancel{e^{i2\pi f_0 t}}+(i2\pi f_0)\gamma C\cancel{e^{i2\pi f_0 t}}+\omega_0^2\cancel{e^{i2\pi f_0 t}}=(F_0/m)\cancel{e^{i2\pi f_0 t}}$$

$$\Rightarrow C=\frac{(F_0/m)}{-4\pi^2 f_0^2+i2\pi f_0\gamma+\omega_0^2}.$$

c) for $t<0$ the solution is $z_1(t)=0$. For $t>0$ the solution is the sum of homogeneous and particular solutions. The parameters A and B are then found by setting $z_1(0^+)=0$ and $z_1'(0^+)=0$.

Case i) $$z_1(t)=\left\{Ae^{-(\frac{1}{2}\gamma-i\sqrt{\omega_0^2-\gamma^2/4})t}+Be^{-(\frac{1}{2}\gamma+i\sqrt{\omega_0^2-\gamma^2/4})t}+\frac{(F_0/m)e^{i2\pi f_0 t}}{-4\pi^2f_0^2+i2\pi f_0\gamma+\omega_0^2}\right\}\text{Step}(t)$$

$$z_1(0^+)=0 \Rightarrow A+B+\frac{(F_0/m)}{-4\pi^2f_0^2+i2\pi f_0\gamma+\omega_0^2}=0$$

$$z_1'(0^+)=0 \Rightarrow -(\tfrac{\gamma}{2}-i\sqrt{\omega_0^2-\gamma^2/4})A-(\tfrac{\gamma}{2}+i\sqrt{\omega_0^2-\gamma^2/4})B+\frac{i2\pi f_0(F_0/m)}{-4\pi^2f_0^2+i2\pi f_0\gamma+\omega_0^2}=0$$

When the above equations are solved for A and B, these parameters will be found as follows:

$$A = -\frac{(F_0/m)}{2\sqrt{\omega_0^2-\gamma^2/4}\left(\frac{i}{2}\gamma+\sqrt{\omega_0^2-\gamma^2/4}-2\pi f_0\right)}$$

$$B = +\frac{(F_0/m)}{2\sqrt{\omega_0^2-\gamma^2/4}\left(\frac{i}{2}\gamma-\sqrt{\omega_0^2-\gamma^2/4}-2\pi f_0\right)}$$

Case ii) $$z_1(t) = \left\{(A+Bt)e^{-\frac{1}{2}\gamma t} + \frac{(F_0/m)\,e^{i2\pi f_0 t}}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2}\right\}\text{step}(t)$$

$$\begin{cases} z_1(0^+) = A + \dfrac{F_0/m}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2} = 0 \Rightarrow A = \dfrac{-F_0/m}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2} \\[2ex] z_1'(0^+) = -\frac{1}{2}\gamma A + B + \dfrac{(F_0/m)(i2\pi f_0)}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2} = 0 \Rightarrow B = \dfrac{-(F_0/m)(i2\pi f_0 + \gamma/2)}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2} \end{cases}$$

Case iii) $$z_1(t) = \left\{A e^{-\left(\frac{\gamma}{2}-\sqrt{\gamma^2/4-\omega_0^2}\right)t} + B e^{-\left(\frac{\gamma}{2}+\sqrt{\gamma^2/4-\omega_0^2}\right)t} + \frac{(F_0/m)\,e^{i2\pi f_0 t}}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2}\right\}\text{step}(t)$$

$$\begin{cases} z_1(0^+) = 0 \Rightarrow A + B + \dfrac{(F_0/m)}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2} = 0 \\[2ex] z_1'(0^+) = 0 \Rightarrow -\left(\frac{\gamma}{2}-\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\right)A - \left(\frac{\gamma}{2}+\sqrt{\gamma^2/4-\omega_0^2}\right)B + \dfrac{i2\pi f_0 (F_0/m)}{-4\pi^2 f_0^2 + i2\pi f_0\gamma + \omega_0^2} = 0 \end{cases}$$

Again, these equations may be solved to yield A and B, as before.